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ON THE ERGODIC THEORY OF NON-INTEGRABLE FUNCTIONS AND INFINITE MEASURE SPACES

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ABSTRACT
We prove a Chow–Robbins type result for an ergodic, non-negative SSSP, and a similar result for transformations preserving infinite measure, which implies that for these transformations, no "absolute" version of Hopf's theorem can hold

§0. Introduction

Let $(\Omega, \mathcal{A}, P, \sigma)$ be an ergodic measure preserving transformation (e.m.p.t.) of a probability space. Birkhoff's pointwise ergodic theorem ([2], ch. 4) states:

THEOREM A. *If $f \in L^1(\Omega, \mathcal{A}, P)$ then $(1/n)\Sigma_{k=0}^{n-1} f \circ \sigma^k \to \int_\Omega f dP$ a.e.*

Theorem A already implies a primitive converse of itself:

PROPOSITION B. *Let $f: \Omega \to \mathbf{R}$ be measurable and non-negative. Then if $\overline{\lim}_{n\to\infty}(1/n)\Sigma_{k=0}^{n-1} f \circ \sigma^k < \infty$ on a set of positive measure then $\int_\Omega f dP < \infty$.*

In [1], Chow and Robbins proved the following converse to the strong law of large numbers:

THEOREM C. *If $\{X_n\}$ are independent, identically distributed random variables and b_n are constants, such that $(1/b_n)\Sigma_{k=1}^{n} X_k \to 1$ a.e. then $E(|X_1|) < \infty$.*

In fact, they proved the stronger result:

THEOREM D. *If $\{X_n\}$ are independent, identically distributed random variables such that $E(|X_1|) = \infty$ then $\forall \{b_n\} \subseteq \mathbf{R}$ either $\overline{\lim}_{n\to\infty}|(1/b_n)\Sigma_{k=1}^{n} X_k| = \infty$ a.e. or $\lim_{n\to\infty}|(1/b_n)\Sigma_{k=1}^{n} X_k| = 0$ a.e.*

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We will "extend" Theorem D to the case where $\{X_n\}$ is an ergodic, non-negative, strictly stationary stochastic process. We prove:

THEOREM 1. *Let* $(\Omega, \mathcal{A}, P, \sigma)$ *be an e.m.p.t., and let* $f: \Omega \to \boldsymbol{R}$ *be measurable and non-negative. If* $E(f) = \infty$ *then* $\forall \{b_n\} \subseteq (0, \infty)$ *either* $\overline{\lim}_{n \to \infty} (1/b_n) \Sigma_{k=0}^{n-1} f \circ \sigma^k = \infty$ *a.e. or* $\lim_{n \to \infty} (1/b_n) \Sigma_{k=0}^{n-1} f \circ \sigma^k = 0$ *a.e. (or both).*

From Theorem 1, we immediately obtain the following converse to Theorem A:

COROLLARY 1. *If* $f: \Omega \to \boldsymbol{R}$ *is measurable and non-negative, and* b_n *are constants s.t.* $(1/b_n) \Sigma_{k=0}^{n-1} f \circ \sigma^k \to 1$ *a.e. then* $E(f) < \infty$.

An adaption of an example of D. Tanny ([4]) will show that Corollary 1 may fail when the assumption of non-negativity is dropped.

Theorem 1 is a consequence of an analogous theorem for conservative e.m.p.t.'s (c.e.m.p.t.'s) of infinite measure spaces. Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. of a non-atomic, σ-finite, infinite measure space. The classic ergodic theorem of Hopf ([2], ch. 4) states that:

THEOREM E. *If* $f, g, \in L'(X, \mathcal{B}, \mu)$ *and* $\int_X g d\mu \neq 0$ *then*

$$\frac{\sum_{k=0}^{n-1} f(T^k x)}{\sum_{k=0}^{n-1} g(T^k x)} \to \frac{\int_X f d\mu}{\int_X g d\mu} \quad \textit{for a.e.} \quad x \in X.$$

It is natural to ask whether an "absolute" version of this can hold, i.e., can there exist constants $\{a_n\}$ such that for every $f \in L^1_+$.

$$(*) \qquad \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \to \int_X f d\mu \quad \text{a.e.}$$

We show that $(*)$ cannot hold even for a single $f \in L^1_+$, and indeed when $\mu(X) = \infty$:

THEOREM 2. *If* $(X, \mathcal{B}, \mu, T)$ *is a c.e.m.p.t. of a non-atomic,* σ*-finite, infinite measure space, then* $\forall \{a_n\} \subseteq \boldsymbol{R}_+$ *either*

$$\overline{\lim_{n \to \infty}} \frac{1}{a_n} \sum_{k=0}^{n-1} p \circ T^k = \infty \quad \textit{a.e.} \quad \forall p \in L^1,\ p \geqq 0,\ \int_X p d\mu > 0$$

or

$$\varliminf_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^{n-1} p \circ T^k = 0 \quad a.e. \quad \forall p \in L^1,\ p \geqq 0$$

(*or both*).

We prove Theorem 2 in §1, and deduce Theorem 1 from it in §2.

We note here that there are c.e.m.p.t.'s $(X, \mathcal{B}, \mu, T)$ of non-atomic, σ-finite, infinite measure spaces together with constants $\{a_n\}$ such that

$$\nu\left(\left\{x : \left|\frac{1}{a_n}\sum_{k=0}^{n-1} f(T^k x) - \int_X f d\mu\right| \geqq \varepsilon\right\}\right) \to 0$$

$\forall f \in L^1$, $\varepsilon > 0$; $\nu \ll \mu$ s.t. $\nu(X) < \infty$.

We will discuss this kind of phenomenon in a later publication, confining ourselves here to pointwise results.

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§1. Proof of Theorem 2

First, we state the equivalent form of Theorem 2 which we prove:

THEOREM 2′. *Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. of a σ-finite, non-atomic measure space. If $\exists \{a_n\} \subseteq \mathbf{R}_+$, and $p, q \in L^1(X, \mathcal{B}, \mu)$ s.t. $p, q \geqq 0$, $\int_X q d\mu > 0$ and*

(i) $\mu(\{x : \varliminf_{n\to\infty} (1/a_n) \sum_{k=0}^{n-1} p(T^k x) > 0\}) > 0$,

(ii) $\mu(\{x : \varlimsup_{n\to\infty} (1/a_n) \sum_{k=0}^{n-1} q(T^k x) < \infty\}) > 0$,

then $\mu(X) < \infty$.

The main idea of the proof, which proceeds in a sequence of steps, is to choose an appropriate set of positive, finite measure, and to show that the return time function of this set has finite integral on it.

STEP 1. No generality is lost in assuming that $a_n < a_{n+1} \uparrow \infty$.

PROOF. For $p \in L^1_+ = \{f \in L^1(X, \mathcal{B}, \mu) : f \geqq 0, \int_X f d\mu > 0\}$. Let $\alpha(p, x) = \varliminf_{n\to\infty} (1/a_n) \sum_{k=0}^{n-1} p(T^k x)$ and $\beta(p, x) = \varlimsup_{n\to\infty} (1/a_n) \sum_{k=0}^{n-1} p(T^k x)$. By conservativity and (ii) $a_n \to \infty$, so

$$\alpha(p, Tx) = \varliminf_{n\to\infty} \frac{1}{a_n}\left[\sum_{k=0}^{n-1} p(T^k x) + p(T^n x) - p(x)\right] \geqq \alpha(p, x)$$

and similarly $\beta(p, Tx) \geqq \beta(p, x)$.

So, by conservativity and ergodicity, $\alpha(p,x)=\alpha(p)$, $\beta(p,x)=\beta(p)$ a.e. Moreover, by Theorem E, $\exists\alpha,\beta\in[0,\infty]$ s.t.

$$\alpha(p)=\alpha\int_X p\,d\mu\,;\ \beta(p)=\beta\int_X p\,d\mu \qquad \forall p\in L^1_+. \tag{1.1}$$

So the hypotheses imply that $0<\alpha\leqq\beta<\infty$.

Now, choose $p\in L^1_+$, $p(x)>0$, $\forall x\in X$ and choose x_0 s.t.

(i) $\Sigma_{n=0}^{\infty}p(T^kx_0)=\infty$,

(ii) $\underline{\lim}_{n\to\infty}(1/a_n)\Sigma_{k=0}^{n-1}p(T^kx_0)=\alpha(p)>0$,

(iii) $\overline{\lim}_{n\to\infty}(1/a_n)\Sigma_{k=0}^{n-1}p(T^kx_0)=\beta(p)<\infty$.

Let $\bar a_n=\Sigma_{k=0}^{n-1}p(T^kx_0)$, then $\bar a_n<\bar a_{n+1}\uparrow\infty$, and $\forall q\in L^1_+$.

$$\underline{\lim}_{n\to\infty}\frac{1}{\bar a_n}\sum_{k=0}^{n-1}q\circ T^k\geqq\frac{\alpha(q)}{\beta(p)}>0,$$

$$\overline{\lim}_{n\to\infty}\frac{1}{\bar a_n}\sum_{k=0}^{n-1}q\circ T^k\leqq\frac{\beta(q)}{\alpha(p)}<\infty. \qquad \square$$

From Step 1 and (1.1) we can choose a_n s.t. $a_n<a_{n+1}\uparrow\infty$ and

$$\frac{\mu(A)}{M}<\underline{\lim}_{n\to\infty}\frac{1}{a_n}\sum_{k=0}^{n-1}1_A\circ T^k\leqq\overline{\lim}_{n\to\infty}\frac{1}{a_n}\sum_{k=0}^{n-1}1_A\circ T^k<\mu(A)\quad\text{a.e.} \tag{1.2}$$

$\forall A\in\mathcal{B}$, $0<\mu(A)<\infty$, where $M\in\mathbf{N}$.

Choose a strictly increasing, continuous function $a(x)$ s.t. $a(n)=a_n$, and let $b(x)$ be the inverse of $a(x)$. Then $b(x)$ is strictly increasing to ∞.

Choose $A\in\mathcal{B}$ such that $\mu(A)=2$. By (1.2), $\exists B\subseteq A$ such that $\mu(B)=1$, and

$$\frac{1}{a(n)}\sum_{k=0}^{n-1}1_A(T^kx)\geqq\delta>0 \qquad \forall n\geqq1,\quad x\in B. \tag{1.3}$$

Let $\varphi(x)=\inf\{k\geqq1\colon T^kx\in B\}$ for $x\in B$ (the return time function of the set B). ($\varphi(x)<\infty$ a.e. by conservativity.)

Define the transformation induced by T on B ([3]) by $T_Bx=T^{\varphi(x)}x$. Let $\varphi_n(x)=\Sigma_{k=0}^{n-1}\varphi(T_B^kx)$. Then ([3]) T_B is an e.m.p.t. and $T_B^nx=T^{\varphi_n(x)}(x)$. By Kac's formula $\mu(X)=\int_B\varphi d\mu$. Let $c(t)=\mu_B(\varphi\geqq t)$, $L(t)=\int_0^t c(y)dy$ (where $\mu_B(C)=\mu(B\cap C)/\mu(B)$).

We will show that $\int_0^\infty(c(t)/L(t))dt<\infty$ which implies that $\int_B\varphi d\mu<+\infty$ or $\mu(X)<+\infty$.

STEP 2. $\Sigma_{n=1}^{\infty}c(b(n))<\infty$.

PROOF.

$$\frac{1}{n}\sum_{j=0}^{\varphi_n(x)-1} 1_A(T^jx) = \frac{\sum_{j=0}^{\varphi_n(x)-1} 1_A(T^j(x))}{\sum_{j=0}^{\varphi_n(x)-1} 1_B(T^jx)} \to 2 \quad \text{a.e. by Theorem E.}$$

Also

$$\sum_{j=0}^{\varphi_n(x)-1} 1_A(T^jx) = \sum_{k=0}^{n-1}\sum_{j=\varphi_k(x)}^{\varphi_{k+1}(x)-1} 1_A(T^jx)$$
$$= \sum_{k=0}^{n-1}\sum_{j=0}^{\varphi(T_B^kx)-1} 1_A(T^j(T_B^kx))$$
$$\geqq \delta\sum_{k=0}^{n-1} a(\varphi(T_B^kx)) \quad \text{by} \quad (1.3).$$

Hence $\overline{\lim}_{n\to\infty}(1/n)\sum_{k=0}^{n-1} a(\varphi(T_B^kx)) < \infty$ a.e. on B. So by Proposition B:

$$\int_B a(\varphi)d\mu < \infty$$
$$\Rightarrow \sum_{n=1}^{\infty} \mu_B(a(\varphi) \geqq n) < \infty$$
$$\Rightarrow \sum_{n=1}^{\infty} \mu_B(\varphi \geqq b(n)) < \infty. \qquad \square$$

Before we continue we need a lemma.

LEMMA. $\forall m \geqq 1\ \exists n_0(m)$ *s.t.* $\forall n \geqq n_0(m)$
(i) $b(mn) \leqq mb(Mn)$,
(ii) $mb(n) \leqq b(Mmn)$.

PROOF. From (1.2)

$$1 < \underline{\lim}_{n\to\infty}\frac{a(\varphi_n)}{n} \leqq \overline{\lim}_{n\to\infty}\frac{a(\varphi_n)}{n} < M \qquad \text{a.e.} \tag{1.4}$$

Hence $\forall m \geqq 1\ \exists n_0(m)$ s.t. $\forall n \geqq n_0(m)\ \exists x \in B$ s.t.

$$n \leqq a(\varphi_n(T_B^{kn}x)) \leqq Mn \quad \text{for} \quad 0 \leqq k \leqq m-1,$$
$$mn \leqq a(\varphi_{mn}(x)) \leqq Mmn.$$

Applying b to these inequalities, and noting that $\varphi_{mn}(x) = \sum_{k=0}^{m-1}\varphi_n(T_B^{kn}x)$ yields (i) and (ii). $\square$

Let $\hat{b}(n) = \sum_{k=0}^{n-1} L(h(k))$.

STEP 3. $\underline{\lim}_{n\to\infty}(\hat{b}(n)/b(n))\geqq 1$.

PROOF. From (1.4) we have that

$$\underline{\lim}_{n\to\infty}\frac{\varphi_n}{b(n)}\geqq 1 \qquad \text{a.e.} \tag{1.5}$$

Let

$$f_n(x)=\begin{cases}\varphi(x) & \text{if} \quad \varphi(x)\leqq b(n)\\ 0 & \text{else.}\end{cases}$$

By Step 2, for a.e. $x\in B$, $\varphi(T_B^k x)\geqq b(k)$ for only finitely many $k\geqq 1$. Hence $f_k(T_B^k x)\neq\varphi(T_B^k x)$ for only finitely many $k\geqq 1$. This, in conjunction with (1.5), gives

$$\underline{\lim}_{n\to\infty}\frac{1}{b(n)}\sum_{k=0}^{n-1}f_k(T_B^k x)\geqq 1 \qquad \text{a.e.,}$$

which in turn, in conjunction with Fatou's lemma, gives

$$\underline{\lim}_{n\to\infty}\frac{1}{b(n)}\sum_{k=0}^{n-1}\int_B f_k d\mu \geqq 1. \tag{1.6}$$

Now

$$\begin{aligned}\int_B f_k d\mu &= \int_0^\infty \mu_B(f_k\geqq y)dy\\ &= \int_0^{b(k)}\mu_B(y\leqq\varphi\leqq b(k))dy\\ &\leqq L(b(k)).\end{aligned}$$

Applying this inequality to (1.6) proves Step 3. □

STEP 4. $\sum_{n=1}^\infty c(\hat{b}(n))<\infty$.

PROOF. Using (ii) of the Lemma with $m=2$ we obtain n_0 s.t. $\forall n\geqq n_0$ $b(n)\leqq\frac{1}{2}b(2Mn)$.

Using Step 3, we obtain $n_1\geqq n_0$ s.t. $\hat{b}(2Mn)\geqq\frac{1}{2}b(2Mn)$, $\forall n\geqq n_1$. Hence $\hat{b}(2Mn)\geqq b(n)$, $\forall n\geqq n_1$. So $\sum_{n=0}^\infty c(\hat{b}(2Mn))<\infty$. But $c\downarrow$ and $b\uparrow$, so this is enough. □

From Step 4 and the definition of $\hat{b}(n)$ we have that

$$\sum_{n=1}^\infty(\hat{b}(n+1)-\hat{b}(n))\frac{c(\hat{b}(n))}{L(b(n))}<\infty. \tag{1.7}$$

By Step 3, for n large

$$\begin{aligned} \hat{b}(n) &\geqq \tfrac{1}{2}b(n) \\ &\Rightarrow b(n) \leqq 2\hat{b}(n) \\ &\Rightarrow L(b(n)) \leqq L(2\hat{b}(n)) \leqq 2L(\hat{b}(n)). \end{aligned}$$

So by (1.7)

$$\sum_{n=1}^{\infty} (\hat{b}(n+1)-\hat{b}(n)) \frac{c(\hat{b}(n))}{L(\hat{b}(n))} < \infty.$$

Hence

$$\begin{aligned} \int_{b(1)}^{\infty} \frac{c(t)}{L(t)}\,dt &= \sum_{n=1}^{\infty} \int_{\hat{b}(n)}^{\hat{b}(n+1)} \frac{c(t)}{L(t)}\,dt \\ &\leqq \sum_{n=1}^{\infty} \int_{\hat{b}(n)}^{\hat{b}(n+1)} \frac{c(\hat{b}(n))}{L(\hat{b}(n))}\,dt \\ &= \sum_{n=1}^{\infty} (\hat{b}(n+1)-\hat{b}(n)) \frac{c(\hat{b}(n))}{L(\hat{b}(n))} < \infty. \end{aligned}$$

Hence

$$\mu(X) = \int_B \varphi d\mu = \int_0^{\infty} c(t)dt = L(\hat{b}(1))\exp\left(\int_{\hat{b}(1)}^{\infty} c(t)/L(t)dt\right) < \infty.$$

Theorem 2 is established. □

We have actually proved (on application of Theorem E):

THEOREM 2″. *If $(X, \mathcal{B}, \mu, T)$ is a c.e.m.p.t. of a non-atomic, σ-finite, infinite measure space then*

$$\forall \{a_n\} \subseteq \boldsymbol{R} \quad \text{and} \quad f \in L^1(X, \mathcal{B}, \mu) \quad \text{s.t.} \quad \int_X f d\mu \neq 0:$$

either

$$\varlimsup_{n\to\infty} \left| \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \right| = \infty \qquad a.e.$$

or

$$\varliminf_{n\to\infty} \left| \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \right| = 0 \qquad a.e.$$

We do not know if Theorem 2″ is true for $f \in L^1$ with $\int_X f d\mu = 0$.

§2. Proof of Theorem 1 and an example

If the conclusion to Theorem 1 fails, then $f \not\equiv 0$ and $b_n \to \infty$. It follows that

$$\underline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k(x) \leqq \underline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k(\sigma x)$$

and

$$\overline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k(x) \leqq \overline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k(\sigma x)$$

thus

$$\underline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k \quad \text{and} \quad \overline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k$$

are constants a.e.; and the failure of the conclusion of Theorem 1 is tantamount to:

(2.1)
$$0 < \alpha = \underline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k(x) \leqq \overline{\lim}_{n\to\infty} \frac{1}{b_n} \sum_{k=0}^{n-1} f \circ \sigma^k(x) = \beta < \infty \qquad \forall x \in \Omega'$$

where $P(\Omega') = 1$. If it is assumed that $E(f) = \infty$, then $b_n/n \to \infty$, and so (2.1) is unchanged by the addition of an integrable function to f. So no generality is lost in assuming $f: \Omega \to \boldsymbol{N}$.

Choose $x_0 \in \Omega'$ and let $\bar{b}_n = \sum_{k=0}^{n-1} f(\sigma^k x_0)$, then $\bar{b}_n < \bar{b}_{n+1}$ and

$$\underline{\lim}_{n\to\infty} \frac{1}{\bar{b}_n} \sum_{k=0}^{n-1} f \circ \sigma^k \geqq \frac{\alpha}{\beta} > 0 \qquad \text{a.e.,}$$

$$\overline{\lim}_{n\to\infty} \frac{1}{\bar{b}_n} \sum_{k=0}^{n-1} f \circ \sigma^k \leqq \frac{\beta}{\alpha} < \infty \qquad \text{a.e.}$$

Normalizing $\bar{b}_n$ we have established:

STEP 1. If the conclusion of Theorem 1 fails, and $E(f) = \infty$, then no generality is lost in assuming:

(i) $f: \Omega \to \boldsymbol{N}$,

(ii) $b_n < b_{n+1}$,

(iii) $1 < \underline{\lim}_{n\to\infty} (1/b_n) \sum_{k=0}^{n-1} f \circ \sigma^k \leqq \overline{\lim}_{n\to\infty} (1/b_n) \sum_{k=0}^{n-1} f \circ \sigma^k < M$ a.e. ($M \in \boldsymbol{N}$).

Choose a continuous, strictly increasing function $b(t)$ s.t. $b(n) = b_n$. Let a be the inverse of b. Then $a(t)$ is continuous and strictly increasing.

Before continuing, we need a lemma, analogous to that of §1:

LEMMA. $\forall m \geqq 1\ \exists n_0$ *s.t.* $\forall n \geqq n_0$:
(i) $b(mn) \leqq Mmb(n)$,
(ii) $mb(n) \leqq Mb(mn)$.

PROOF. By (iii) of Step 1, $\forall n \geqq 1\ \exists n_0(m)$ s.t. $\forall n \geqq n_0(m)\ \exists x \in \Omega$ s.t.

$$b(n) \leqq \sum_{k=0}^{n-1} f \circ \sigma^k(\sigma^{jn}x) \leqq Mb(n) \qquad 0 \leqq j \leqq m-1$$

$$b(mn) \leqq \sum_{k=0}^{mn-1} f(\sigma^k x) \leqq Mb(mn).$$

The inequalities follow from the fact that

$$\sum_{k=0}^{mn-1} f(\sigma^k x) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f \circ \sigma^k(\sigma^{jn}x). \qquad \square$$

Let $f_n(x) = \sum_{k=0}^{n-1} f \circ \sigma^k(x)$.

STEP 2.

$$1 \leqq \varliminf_{n\to\infty} \frac{a(f_n)}{n} \leqq \varlimsup_{n\to\infty} \frac{a(f_n)}{n} \leqq M^2 \qquad \text{a.e.}$$

PROOF. From (ii) of the Lemma, with $m = M^2$, we get n_0 s.t. $\forall n \geqq n_0$,

$$M^2 b(n) \leqq Mb(M^2 n), \quad \text{i.e.} \quad Mb(n) \leqq b(M^2 n).$$

Now, by (iii) of Step 1, for a.e. $x \in \Omega\ \exists N$ s.t. $\forall n \geqq N$

$$b(n) \leqq f_n \leqq Mb(n) \leqq b(M^2 n)$$

$$\Rightarrow n \leqq a(f_n) \leqq M^2 n. \qquad \square$$

Now we build a tower on $(\Omega, \mathcal{A}, P, \sigma)$ with f as the height function ([3]):

$X = \{(x, n) : f(x) \geq n,\ n \geqq 1\}$,
$(A, n) = \{(x, n) : x \in A\}$ where $A \in \mathcal{A} \cap [f \geqq n]$;

$$\mathcal{B} = \bigvee_{n=1}^{\infty} (\mathcal{A} \cap [f \geqq n], n),\ \mu = \sum_{n=1}^{\infty} P|_{(\mathcal{A} \cap [f \geqq n], n)},$$

$$T(x, n) = \begin{cases} (x, n+1) & \text{if} \quad f(x) \geqq n+1 \\ (\sigma x, 1) & \text{if} \quad f(x) = n. \end{cases}$$

It was shown in [3] that $(X, \mathcal{B}, \mu, T)$ is a c.e.m.p.t. and that $\mu(X) = E(f)$. We derived Step 2 on the assumptions that $E(f) = \infty$ and that the theorem fails. We prove the theorem by deriving from Step 2 that $\mu(X) < \infty$ (contradicting the assumption that $E(f) = \infty$). Let $\bar{\Omega} = (\Omega, 1)$. Clearly

$$\sum_{j=0}^{f_n(x)-1} 1_{\bar{\Omega}}(T'(x,1)) = n \qquad \forall x \in \Omega.$$

Choose $k_n(x)$ s.t. $f_{k_n(x)}(x) \leqq n < f_{k_n(x)+1}(x)$. Then $k_n(x) \to \infty$ a.e. and

$$\begin{aligned}\underline{\lim}_{n\to\infty} \frac{1}{a(n)} \sum_{k=0}^{n-1} 1_{\bar{\Omega}}(T^k(x,1)) &\geqq \underline{\lim}_{n\to\infty} (f_{k_n+1}(x)) \sum_{j=0}^{f_{k_n}-1} 1_{\Omega}(T'(x,1)) \\ &= \underline{\lim}_{n\to\infty} \frac{k_n(x)}{a(f_{k_n+1}(x))} \geqq \frac{1}{M^2} \qquad \text{by Step 2,}\end{aligned}$$

$$\begin{aligned}\overline{\lim}_{n\to\infty} \frac{1}{a(n)} \sum_{k=0}^{n-1} 1_{\bar{\Omega}}(T^k(x,1)) &\leqq \overline{\lim}_{n\to\infty} \frac{1}{a(f_{k_n}(x))} \sum_{j=0}^{f_{k_n+1}-1} 1_{\bar{\Omega}}(T'(x,1)) \\ &= \overline{\lim}_{n\to\infty} \frac{k_n(x)+1}{a(f_{k_n}(x))} \leqq 1 \qquad \text{by Step 2.}\end{aligned}$$

So by Theorem 2 $\mu(X) < \infty$. □

An Example. We construct an e.m.p.t. $(\Omega, \mathscr{A}, P, \sigma)$ and a measurable $X : \Omega \to \boldsymbol{Z}$ s.t. $E(|X|) = \infty$ and $(1/n)\Sigma_{k=0}^{n-1} X \circ \sigma^k \to 1$ a.e.

Construction 1 (*Ex.* (a), [4]). An e.m.p.t. $(\bar{\Omega}, \bar{\mathscr{A}}, \bar{P}, \bar{\sigma})$ and a measurable $f : \bar{\Omega} \to \boldsymbol{N}$ s.t. $E(f) = \infty$ and $(1/n) f \circ \bar{\sigma}^n \to 0$ a.e.

Let $(\Omega', \mathscr{A}', P', \sigma')$ be an e.m.p.t., let $\varphi : \Omega' \to \boldsymbol{N}$ be s.t. $E(\varphi) < \infty$, $E(\varphi^2) = \infty$. We define f on the tower above σ' with height function φ.

Let $\bar{\Omega} = \{(x,n) : n \geqq 1, \varphi(x) \geqq n\}$, $\bar{\mathscr{A}} = \vee_{n=1}^{\infty} (\mathscr{A}' \cap [\varphi \geqq n], n)$, $\bar{P} = (1/C)\Sigma_{n=1}^{\infty} P'|_{(\mathscr{A}' \cap [\varphi \geqq n], n)}$ where $C = E(\varphi)$,

$$\bar{\sigma}(x,n) = \begin{cases} (x, n+1) & \text{if} \quad \varphi(x) \geqq n+1 \\ \\ (\sigma' x, 1) & \text{if} \quad \varphi(x) = n. \end{cases}$$

Then ([3]) $(\bar{\Omega}, \bar{\mathscr{A}}, \bar{P}, \bar{\sigma})$ is an e.m.p.t. and $\bar{P}(\bar{\Omega}) = 1$.

Let $f : \bar{\Omega} \to N$ be defined by $f(x,n) = n$;

$$E(f) = \frac{1}{C} \sum_{k=1}^{\infty} k P'(\varphi \geqq k) = \frac{1}{C} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} P'(\varphi = n) = \infty.$$

If we show that $f \circ (\bar{\sigma}^n(x,1))/n \to 0$ for a.e. $x \in \Omega'$ then by the ergodicity of $\bar{\sigma}$, $f \circ \bar{\sigma}^n / n \to 0$ a.e. on $\bar{\Omega}$.

Let $\varphi_n(x) = \Sigma_{k=0}^{n-1} \varphi(\sigma'^k x)$. If $\varphi_k(x) \leqq n < \varphi_{k+1}(x)$ then $f(\bar{\sigma}^n(x,1)) \leqq \varphi(\sigma'^k x)$

and so $f(\bar{\sigma}^n(x,1)/n \leqq \varphi(\sigma'^k x)/\varphi_k(x)$. By the ergodic theorem $\varphi(\sigma'^k x)/\varphi_k(x) \to 0$ a.e. on Ω' as $k \to \infty$. So construction 1 is finished. □

We now construct the Example.

Construction 2 (*Adapted from Ex.* (b), [4]). Let $I = [0,1)$, $\mathcal{B}$ be the Borel σ-algebra, λ be Lebesgue measure and $\tau x = 2x \mod 1$. Let

$$u(x) = \begin{cases} -1 & \text{if } x \in [0, \frac{1}{2}) \\ \\ +1 & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

then the $\{u \circ \tau^n\}$ are independent, identically distributed random variables, and $(I, \mathcal{B}, \lambda, \tau)$ is mixing.

Let $\Omega = \bar{\Omega} \times I$, $\mathcal{A} = \bar{\mathcal{A}} \times \mathcal{B}$, $P = \bar{P} \times \lambda$ and $\sigma = \bar{\sigma} \times \tau$, then $(\Omega, \mathcal{A}, P, \sigma)$ is e.m.p.t.

Let $X(x,y) = f(x)u(y) - f(\bar{\sigma}x)u(\tau y) + 1$, then

$$E(X^+) \geqq \int_{\bar{\Omega} \times [u=1, u \circ \tau = -1]} (f(x) + f(\bar{\sigma}x) + 1)dP(x,y) = \frac{1}{4} \int_{\bar{\Omega}} (f + f \circ \bar{\sigma} + 1)dP = \infty$$

and

$$\frac{1}{n} \sum_{k=0}^{n-1} X \circ \sigma^k(x,y) = 1 + \frac{f(x)u(y) - f(\bar{\sigma}^n x)u(\tau^n y)}{n} \to 1 \qquad \text{a.e.} \qquad \square$$

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